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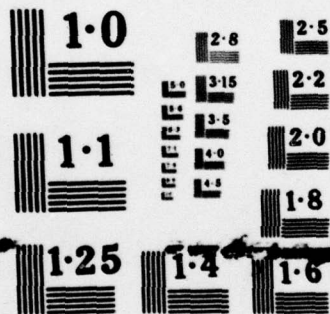UNCLASSIFIED SERIAL-T-393 ARO-13622.7-M NL

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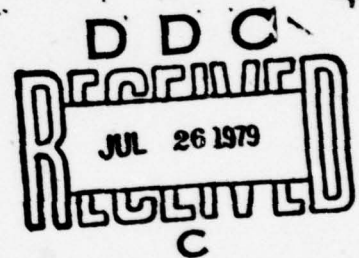SCHOOL OF ENGINEERING
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EXTENSIONS OF THE GAUVIN-TOLLE OPTIMAL VALUE DIFFERENTIAL
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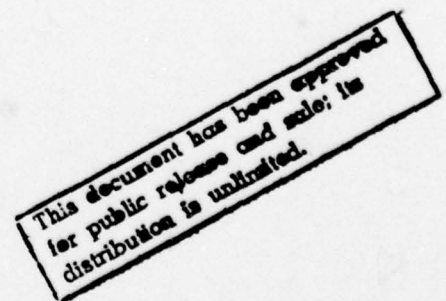
by

Anthony V. Fiacco
William P. Hutzler

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Gauvin and Tolle have obtained bounds on the directional derivative limit quotient of the optimal value function for mathematical programs containing a right-hand side perturbation. In this paper, we extend the results of Gauvin and Tolle to the general mathematical program in which a parameter appears arbitrarily in the constraints and in the objective function. An implicit function theorem is applied to transform the general

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20. Abstract (continued)

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EXTENSIONS OF THE GAUVIN-TOLLE OPTIMAL VALUE DIFFERENTIAL
STABILITY RESULTS TO GENERAL MATHEMATICAL PROGRAMS

by

Anthony V. Fiacco *
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Abstract

Gauvin and Tolle have obtained bounds on the directional deriva-
tive limit quotient of the optimal value function for mathematical
programs containing a right-hand side perturbation. In this paper,
we extend the results of Gauvin and Tolle to the general mathematical
program in which a parameter appears arbitrarily in the constraints
and in the objective function. An implicit function theorem is
applied to transform the general mathematical program to a locally
equivalent inequality constrained program, and, under conditions
used by Gauvin and Tolle, their upper and lower bounds on the optimal
value function directional derivative limit quotient are shown to
pertain to this reduced program. These bounds are then shown to apply
in programs having both inequality and equality constraints where a
parameter may appear anywhere in the program.

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EXTENSIONS OF THE GAUVIN-TOLLE OPTIMAL VALUE DIFFERENTIAL
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1. Introduction

The sensitivity of the optimal value function of a mathematical
program to perturbations of the problem parameters has been addressed
by a number of authors. Using point-to-set maps, Berge [4] derived
conditions sufficient for the semi-continuity of the optimal value
function for programs with constraint set perturbations, and provided
a general framework for some of the earliest work on the variation of
the "perturbation function," i.e., the optimal objective function
value, with changes in a parameter appearing in the right-hand side of
the constraints. Evans and Gould [7] gave conditions guaranteeing the
continuity of the perturbation function when the constraints are func-
tional inequalities. Greenberg and Pierskalla [13] extended the work
of Evans and Gould to obtain results for general constraint perturba-
tions and obtained some initial results for programs with equality con-
straints. In [15], Hogan established conditions sufficient for the
continuity of the perturbation function of a convex program, and in
[16] gave conditions implying the continuity of the optimal value

function of a non-convex program when a parameter appears in the objective function.

The first- and second-order variation of the optimal value of a general nonlinear program under quite arbitrary parametric perturbations has been investigated by Hogan [15], Armacost and Fiacco [1,2,3], Fiacco [9], and Fiacco and McCormick [10]. In [2] the optimal value function is shown, under strong conditions, to be twice continuously differentiable, with respect to the problem parameters, with its parameter gradient (Hessian) equal to the gradient (Hessian) of the Lagrangian of the problem. Armacost and Fiacco [1] have also obtained first- and second-order expressions for changes in the optimal value function as a function of right-hand side perturbations.

A number of results relating to the differential stability of the optimal value function have also been obtained, generally associated with the existence of directional derivatives or bounds on the directional derivative limit quotient. Danskin [5,6] provided one of the earliest characterizations of the differential stability of the optimal value function of a mathematical program. Addressing the problem minimize $f(x,\varepsilon)$ subject to $x\varepsilon S$, S some topological space, ε in E^k, Danskin derived conditions under which the directional derivative of f^* exists and also determined its representation. This result has wide applicability in the sense that the constraint space, S, can be any compact topological space. However, the result is restricted to a constraint set that does not vary with the parameter ε. For the special case in which S is defined by inequalities involving a parameter, $g_i(x,\varepsilon) \geq 0$ for $i = 1,\ldots,m$, where f is convex

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and the g_i are concave on S, Hogan [15] has given conditions that imply that the directional derivative of f* exists and is finite in all directions.

For programs without equality constraints, Robinson [20] has shown that, under certain second-order conditions, the optimal value function satisfies a stability of degree two. Under this stability property, bounds on the directional derivative of f* can be derived. For convex programming problems, Gol'stein [12] has shown that a saddle point condition is satisfied by the directional derivative of f*. Gauvin and Tolle [11], not assuming convexity, but limiting their analysis to right-hand side perturbations, extended the work of Gol'stein and provide sharp upper and lower bounds on the directional derivative limit quotient of f*, assuming the Mangasarian-Fromovitz constraint qualification and without requiring the existence of second-order conditions. Sensitivity results for infinite dimensional programs have recently been obtained by Maurer [18,19], who developed a representation for the directional derivative of the subgradient of the optimal value function of such problems.

The purpose of this paper is to extend the work of Gauvin and Tolle to the general mathematical program in which a parameter appears arbitrarily in the constraints and the objective function. For this problem we obtain the Gauvin-Tolle upper and lower bounds on the directional derivative limit quotient of the optimal value function. The mode of proof closely parallels that given by Gauvin and Tolle [11].

2. Notation and Definitions

In this paper we shall be concerned with mathematical programs of the form:

$$\min_x f(x,\varepsilon) \qquad\qquad\qquad P(\varepsilon)$$

$$\text{s.t. } g_i(x,\varepsilon) \geq 0 \ (i=1,\ldots,m), \ h_j(x,\varepsilon) = 0 \ (j=1,\ldots,p),$$

where $x \varepsilon E^n$ is the vector of decision variables, ε is a parameter vector in E^k, and the functions f, g_i and h_j are once continuously differentiable on $E^n \times E^k$. The feasible region of problem $P(\varepsilon)$ will be denoted $R(\varepsilon)$ and the set of solutions $S(\varepsilon)$. The m-vector whose components are $g_i(x,\varepsilon)$, $i = 1,\ldots,m$, and the p-vector whose components are $h_j(x,\varepsilon)$, $j = 1,\ldots,p$, will be denoted by $g(x,\varepsilon)$ and $h(x,\varepsilon)$, respectively.

Following usual conventions the gradient, with respect to x, of a once differentiable real-valued function $f:E^n \times E^k \to E^1$ is denoted $\nabla_x f(x,\varepsilon)$ and is taken to be the row vector $[\partial f(x,\varepsilon)/\partial x_1, \ldots, \partial f(x,\varepsilon)/\partial x_n]$. If $g(x,\varepsilon)$ is a vector-valued function, $g:E^n \times E^k \to E^m$, whose components $g_i(x,\varepsilon)$ are differentiable in x, then $\nabla_x g(x,\varepsilon)$ denotes the m x n Jacobian matrix of g whose ith row is given by $\nabla_x g_i(x,\varepsilon)$, $i = 1,\ldots,m$. The transpose of the Jacobian $\nabla_x g(x,\varepsilon)$ will be denoted $\nabla_x' g(x,\varepsilon)$. Differentiation with respect to the vector ε is denoted in a completely analogous fashion. Henceforth, we do not distinguish between row and column vectors in this paper; their use should be clear from the context in which they are applied.

The Lagrangian for $P(\varepsilon)$ will be written

$$L(x,\mu,\omega,\varepsilon) = f(x,\varepsilon) - \sum_{i=1}^{m} \mu_i g_i(x,\varepsilon) + \sum_{j=1}^{p} \omega_j h_j(x,\varepsilon),$$

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and the set of Kuhn-Tucker vectors corresponding to the decision vector x will be given by

$$K(x,\varepsilon) = \{(\mu,\omega)\varepsilon E^m x E^p : \nabla_x L(x,\mu,\omega,\varepsilon) = 0,\ \mu_i \geq 0,\ \mu_i g_i(x,\varepsilon) = 0,\ i=1,\ldots,m\}.$$

Writing a solution vector as a function of the parameter ε, the index set for inequality constraints which are binding at a solution $x(\varepsilon)$ is denoted by $B(\varepsilon) = \{i : g_i(x(\varepsilon),\varepsilon) = 0\}$. Finally, the optimal value function will be defined as

$$f^*(\varepsilon) = \min\ \{f(x,\varepsilon) : x\varepsilon R(\varepsilon)\}.$$

Throughout this paper we shall make use of the well known Mangasarian-Fromovitz Constraint Qualification (MFCQ) which holds at at point $x\varepsilon R(\varepsilon)$ if:

i) there exists a vector $\widetilde{y}\varepsilon E^n$ such that

$$\nabla_x g_i(x,\varepsilon)\ \widetilde{y} > 0 \text{ for i such that } g_i(x,\varepsilon) = 0 \text{ and} \quad (2.1)$$

$$\nabla_x h_j(x,\varepsilon)\ \widetilde{y} = 0 \text{ for } j=1,\ldots,p; \text{ and} \quad\quad\quad (2.2)$$

ii) the gradients $\nabla_x h_j(x,\varepsilon)$, $j=1,\ldots,p$, are linearly independent.

We will also have occasion to make use of the notions of semi-continuity for both real-valued functions and point-to-set maps. There are several equivalent definitions for these properties. The ones most suited to our purpose are given below. The reader interested in a more complete development of these properties is referred to Berge [4] and Hogan [17].

Definition 2.1. Let ϕ be a real-valued function defined on the space X.

    i) ϕ is said to be lower semi-continuous at a point $x_o \epsilon X$ if

$$\lim_{x \to x_o} \phi(x) \geq \phi(x_o).$$

    ii) ϕ is said to be upper semi-continuous at a point $x_o \epsilon X$ if

$$\overline{\lim_{x \to x_o}} \phi(x) \leq \phi(x_o).$$

Using these definitions, one readily sees that a real-valued function ϕ is continuous at a point if and only if it is both upper and lower semi-continuous at that point.

Definition 2.2. Let $\phi: X \to Y$ be a point-to-set mapping and let $\{\epsilon_n\} \subset X$ with $\epsilon_n \to \bar{\epsilon}$ ($\bar{\epsilon}$ in X).

    i) ϕ is said to be lower semi-continuous at a point $\bar{\epsilon}$ of X if, for each $\bar{x} \epsilon \phi(\bar{\epsilon})$, there exists a value n_o and a sequence $\{x_n\} \subset Y$ with $x_n \epsilon \phi(\epsilon_n)$ for $n \geq n_o$ and $x_n \to \bar{x}$.

    ii) ϕ is said to be upper semi-continuous at a point $\bar{\epsilon}$ of X if $x_n \epsilon \phi(\epsilon_n)$ and $x_n \to \bar{x}$ together imply that $\bar{x} \epsilon \phi(\bar{\epsilon})$.

Following Berge [4], we denote the lower (upper) semi-continuity of point-to-set maps by l.s.c. (u.s.c.); for real-valued functions we use the notation lsc and usc for lower and upper semi-continuity, respectively.

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<u>Definition 2.3.</u> A point-to-set mapping $\phi:X\rightarrow Y$ is said to be uniformly compact near a point $\bar{\epsilon}$ of X if the closure of the set $\underset{\epsilon \text{ in } N(\bar{\epsilon})}{\cup \phi(\epsilon)}$ is compact for some neighborhood $N(\bar{\epsilon})$ of $\bar{\epsilon}$.

In Section 3 we apply a reduction of variables technique to $P(\epsilon)$ which transforms that program to an equivalent program involving only inequality constraints. This approach simplifies the derivation of intermediate results which are needed to derive the bounds on the directional derivative limit quotients of $f^*(\epsilon)$ given in Section 4. A demonstration of the results is provided in the example of Section 5. Section 6 concludes with a few remarks concerning related results.

3. Reduction of Variables

In $P(\epsilon)$, if the rank of the Jacobian, $\nabla_x h$, with respect to x of the (first n) equality constraints in a neighborhood of a solution is equal to n, then the given solution is completely determined as a solution of the system of equations $h_j(x,\epsilon) = 0$, $j = 1,\ldots,n$, and the (locally unique) solution, $x(\epsilon)$, of this system near $\epsilon = 0$ is then completely characterized by the appropriate implicit function theorem, depending on the assumptions about ϵ, as for example in [9] and [20]. We are here interested in the less structured situation and hence assume that the rank of $\nabla_x h$ is less than n. Since we shall be making use of MFCQ, this entails the assumption that the number p of equality constraints is less than n. If there are no equality constraints in a particular formulation of $P(\epsilon)$, simply suppress reference to h in the following development. Otherwise, we take advantage of the linear independence assumption to eliminate the equalities, again using an implicit function theorem.

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If the mapping $h: E^n \times E^k \to E^p$ satisfies the following conditions:

i) h is continuous on an open neighborhood of the point $(x_D^*, x_I^*, \epsilon^*)$, where $x_D^* \in E^p$, $x_I^* \in E^{n-p}$, and ϵ^* is in E^k,

ii) $h(x_D^*, x_I^*, \epsilon^*) = 0$,

iii) the $p \times p$ Jacobian $\nabla_{x_D} h(x_D, x_I, \epsilon)$ of h exists in a neighborhood of $(x_D^*, x_I^*, \epsilon^*)$ and is continuous at that point, and

iv) $\nabla_{x_D} h(x_D^*, x_I^*, \epsilon^*)$ is nonsingular,

where $x = (x_D, x_I)$, then the usual implicit function theorem results hold, i.e., the system of equations $h(x_D, x_I, \epsilon) = 0$ can be solved for x_D in terms of x_I and ϵ for any x_I and ϵ near x_I^* and ϵ^* respectively. Furthermore, this representation is unique and the resulting function $x_D = x_D(x_I, \epsilon)$ is continuous in a neighborhood, N^*, of (x_I^*, ϵ^*) and $x_D^* = x_D(x_I^*, \epsilon^*)$. Thus, in N^*, the system $h(x_D, x_I, \epsilon) = 0$ is satisfied identically by the function $x_D = x_D(x_I, \epsilon)$. Under our additional assumption that h is once continuously differentiable in x_I and ϵ, $x_D(x_I, \epsilon)$ is also once continuously differentiable in x_I and ϵ.

Applying this result to $P(\epsilon)$, near (x_I^*, ϵ^*), since we have $\widetilde{h}(x_I, \epsilon) \equiv h[x_D(x_I, \epsilon), x_I, \epsilon] = 0$, this problem can be reduced to one involving only inequality constraints:

$$\min_{x_I} \widetilde{f}(x_I, \epsilon)$$

$$\widetilde{P}(\epsilon)$$

$$\text{s.t.} \quad \widetilde{g}_i(x_I, \epsilon) \geq 0 \qquad (i = 1, \ldots, m),$$

where $\widetilde{f}(x_I, \varepsilon) \equiv f[x_D(x_I, \varepsilon), x_I, \varepsilon]$ and $\widetilde{g}_i(x_I, \varepsilon) \equiv g_i[x_D(x_I, \varepsilon), x_I, \varepsilon]$ for $i = 1, \ldots, m$, and where the minimization is now performed over the n-p dimensional vector x_I. The programs P and $\widetilde{P}$ are equivalent in a neighborhood of (x_I^*, ε^*), in the sense that the point $x(\varepsilon) \varepsilon E^n$, with $x(\varepsilon) = (x_D(\varepsilon), x_I(\varepsilon))$, satisfies the Kuhn-Tucker first-order necessary conditions for an optimum of $P(\varepsilon)$ if and only if the point $x_I(\varepsilon)$ satisfies those conditions for $\widetilde{P}(\varepsilon)$, where $x_D(x_I, \varepsilon)$ is as given above.

We now show that the Mangasarian-Fromovitz constraint qualification for $P(\varepsilon)$ is inherited by the reduced problem $\widetilde{P}(\varepsilon)$. For simplicity in notation, and without loss of generality, assume that $\varepsilon^* = 0$, assume that x^* is a local solution of $P(0)$, and assume that the components of x have been relabeled so that $x = (x_D, x_I)$ and $\nabla_{x_D} h(x_D^*, x_I^*, 0)$ is non-singular. We first state, without proof, the straightforwardly proved equivalence result that establishes the connection between local solution points of P and $\widetilde{P}$.

Lemma 3.1. If $f, g, h \in C^1$, and the once continuously differentiable vector function $x_D = x_D(x_I, \varepsilon)$ is given (e.g., by the implicit function theorem as indicated) such that $h(x_D(x_I, \varepsilon), x_I, \varepsilon) \equiv 0$ in a neighborhood of $(x^*, 0) = (x_D^*, x_I^*, 0)$, then near $\varepsilon = 0$, the point $x(\varepsilon)$ satisfies the Kuhn-Tucker first-order necessary conditions for an optimum of $P(\varepsilon)$ if and only if the point $x_I(\varepsilon)$ is a Kuhn-Tucker point of $\widetilde{P}(\varepsilon)$. Furthermore, near $\varepsilon = 0$, $x(\varepsilon)$ is a local solution of $P(\varepsilon)$ if and only if $x_I(\varepsilon)$ is a local solution of $\widetilde{P}(\varepsilon)$, where $x(\varepsilon) = (x_D(x_I(\varepsilon), \varepsilon), x_I(\varepsilon))$.

If there are no inequality constraints in $P(\varepsilon)$, then $\widetilde{P}(\varepsilon)$ is an unconstrained problem. In this instance, restriction of x_I to a compact set containing a solution set of $\widetilde{P}(0)$ makes it possible to obtain

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immediately the directional derivative, $D_z \tilde{f}^*(0)$ (see (4.1) for the definition) of $\tilde{f}^*(0)$, the optimal value of $\tilde{P}(0)$, in any direction z and in terms of x_I, by using a result due to Danskin [5]. Since $x_I = x_D(x_I, \epsilon)$, it is easy to calculate the corresponding result for $P(\epsilon)$ in terms of $x = (x_D, x_I)$. Since this result is readily obtained and since we have not seen this development in the literature, we give the details before analyzing the more difficult situation involving inequalities.

Assume that the feasible region $R(\epsilon)$ of $P(\epsilon)$ is nonempty and uniformly compact near $\epsilon = 0$. Then, the solution sets $S(\epsilon)$ of $P(\epsilon)$ and $\tilde{S}(\epsilon)$ of $\tilde{P}(\epsilon)$ will exist and will be uniformly compact near $\epsilon = 0$. Consider $\tilde{P}(\epsilon)$: min $\tilde{f}(x_I, \epsilon)$ s.t. $x_I \epsilon \tilde{R}$, a nonempty compact set, independent of ϵ, such that for ϵ near 0 the interior of $\tilde{R}$ contains $\tilde{S}(\epsilon)$. Danskin's result (see also (6.1) and the ensuing discussion) says that under the assumed conditions, $D_z \tilde{f}^*(0) = \min_{x_I} z \nabla_\epsilon \tilde{f}(x_I, 0)$ s.t. $x_I \epsilon \tilde{S}(0)$, the set of solution points of $\tilde{P}(0)$. Since $x_D = x_D(x_I, \epsilon)$ is a differentiable function, and since $h[x_D(x_I, \epsilon), x_I, \epsilon] \equiv 0$, it is easy to show that

$$\nabla_\epsilon \tilde{f} = \nabla_{x_D} f \nabla_\epsilon x_D + \nabla_\epsilon f = - \nabla_{x_D} f \nabla_{x_D} h^{-1} \nabla_\epsilon h + \nabla_\epsilon f = \omega \nabla_\epsilon h + \nabla_\epsilon f = \nabla_\epsilon L(x, \omega, \epsilon),$$

where $\omega = - \nabla_{x_D} f \nabla_{x_D} h^{-1}$ and L is the Lagrangian of $P(\epsilon)$ (without the inequality constraints). Since $\tilde{f}^*(\epsilon) \equiv f^*(\epsilon)$, it readily follows that

$$D_z f^*(0) = \min_{x \epsilon S(0)} z \nabla_\epsilon L(x, \omega(x), 0).$$

This result also follows as a specialization of our general results (see Corollary 4.6 and the note following the proof) as does Danskin's result (see the discussion just before and after (6.1)). It may also be of interest to observe that if the transformation

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$x_D = x_D(x_I, \epsilon)$, or any nonsingular transformation, results in a problem whose constraints are not dependent on the parameter ϵ, then the above approach applies, the domain of interest of the transformed problem simply being the intersection of the set of points satisfying the parameter-free constraints with a compact set $\widetilde{R}$ selected as above.

We now turn to the development of the results for the general problem $P(\epsilon)$ where it is assumed that both inequalities and equalities are present.

Theorem 3.2. If $f, g, h \epsilon C^1$, then MFCQ holds for $P(0)$ at $(x^*, 0) = (x_D^*, x_I^*, 0)$ with $\widetilde{y} = (\widetilde{y}_D, \widetilde{y}_I) \epsilon E^n$ the associated vector, where $\widetilde{y}_D \epsilon E^p$ and $\widetilde{y}_I \epsilon E^{n-p}$, if and only if MFCQ holds in problem $\widetilde{P}(0)$ at the point $(x_I^*, 0)$ with vector $\widetilde{y}_I$.

Proof. Suppose that MFCQ holds for $P(0)$ at $(x^*, 0) = (x_D^*, x_I^*, 0)$ with $\widetilde{y} = (\widetilde{y}_D, \widetilde{y}_I)$ the associated vector. Writing $\nabla_x h$ as $\left[\nabla_{x_D} h : \nabla_{x_I} h\right]$, we see that (2.2) can be expressed as:

$$\nabla_{x_D} h(x^*, 0) \widetilde{y}_D + \nabla_{x_I} h(x^*, 0) \widetilde{y}_I = 0. \tag{3.1}$$

Since we have assumed that $\nabla_{x_D} h(x^*, 0)$ is nonsingular, we can solve for $\widetilde{y}_D$ in (3.1) and obtain:

$$\widetilde{y}_D = -\left[\nabla_{x_D} h(x^*, 0)\right]^{-1} \nabla_{x_I} h(x^*, 0) \widetilde{y}_I. \tag{3.2}$$

Now, denoting the inequality constraints of $\widetilde{P}(0)$ by $\widetilde{g}_i$, i.e., $\widetilde{g}_i = g_i(x_D(x_I, 0), x_I, 0)$, $i = 1, \ldots, m$, by differentiating with respect to x_I we obtain:

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$$\nabla_{x_I} \widetilde{g}_i = \nabla_{x_D} g_i \; \nabla_{x_I} x_D + \nabla_{x_I} g_i,$$

or

$$\nabla_{x_I} \widetilde{g}_i = \nabla_x g_i \begin{bmatrix} \nabla_{x_I} x_D \\ I \end{bmatrix}. \tag{3.3}$$

Multiplying by $\widetilde{y}_I$ in (3.3) we have:

$$\nabla_{x_I} \widetilde{g}_i \; \widetilde{y}_I = \nabla_x g_i \begin{bmatrix} \nabla_{x_I} x_D \\ I \end{bmatrix} \widetilde{y}_I. \tag{3.4}$$

But $h(x_D(x_I,0),x_I,0) \equiv 0$ so that

$$\nabla_{x_D} h(x_D(x_I,0),x_I,0) \; \nabla_{x_I} x_D + \nabla_{x_I} h(x_D(x_I,0),x_I,0) = 0,$$

and since $\nabla_{x_D} h(x_D(x_I,0),x_I,0)$ is nonsingular, we obtain:

$$\nabla_{x_I} x_D = - \left[\nabla_{x_D} h(x_D(x_I,0),x_I,0) \right]^{-1} \nabla_{x_I} h(x_D(x_I,0),x_I,0).$$

Substituting this last expression in (3.4) we have:

$$\nabla_{x_I} \widetilde{g}_i \; \widetilde{y}_I = \nabla_x g_i \begin{bmatrix} -[\nabla_{x_D} h(x_D(x_I,0),x_I,0)]^{-1} \nabla_{x_I} h(x_D(x_I,0),x_I,0) \\ I \end{bmatrix} \widetilde{y}_I,$$

and from (3.2) we see that at $(x_I^*,0)$

$$\nabla_{x_I} \widetilde{g}_i \; \widetilde{y}_I = \nabla_x g_i \begin{bmatrix} \widetilde{y}_D \\ \widetilde{y}_I \end{bmatrix} = \nabla_x g_i \; \widetilde{y}. \tag{3.5}$$

Thus, by (2.1) it follows that $\nabla_{x_I} \widetilde{g}_i \; \widetilde{y}_I > 0$.

To prove the converse, we first note that, by the hypotheses of the implicit function theorem, $\nabla_{x_D} h(x_D^*, x_I^*, 0)$ is nonsingular. Thus, the gradients $\nabla_x h_j(x^*, 0)$, $j = 1, \ldots, p$, are linearly independent and by choosing

$$\widetilde{y}_D = -\left[\nabla_{x_D} h(x^*, 0)\right]^{-1} \nabla_{x_I} h(x^*, 0) \, \widetilde{y}_I,$$

(2.2) is satisfied for $j = 1, \ldots, p$. The inequality in (2.1) now follows from (3.5), a direct consequence of (3.1).

In [11], Gauvin and Tolle established that the set of Kuhn-Tucker multipliers associated with a solution, x^*, of $P(0)$ is non-empty, compact and convex if and only if MFCQ is satisfied at x^*. That result enables us to establish in Theorem 3.3, a necessary link between a directional derivative, with respect to the decision variable x_I, of the objective function at an optimal point and a directional derivative of the Lagrangian taken with respect to the parameter ε. It is this relationship which eventually leads to the upper and lower bounds on the directional derivative of $\widetilde{f}^*$ which are derived in the next section.

For notational simplicity, throughout the remainder of this paper we shall refer to the problem functions of the program $\widetilde{P}(\varepsilon)$ without the 'tilda' notation. The distinction between reference to $P(\varepsilon)$ and $\widetilde{P}(\varepsilon)$ should be clear from the text. Also, the decision variables for the reduced problem will not be subscripted, and, unless specified to the contrary, all gradients will be understood to be taken with respect to the relevant decision variable.

The next two theorems are crucial in obtaining the sharp bounds on the optimal value directional derivative limit quotient. They show that, at a local minimum where MFCQ holds, there exists a direction (in E^{n-p}) in which the directional derivative of the objective function yields that portion of the bound attributable to the constraint perturbation.

<u>Theorem 3.3</u>. If the conditions of MFCQ are satisfied for some $x^* \epsilon \widetilde{S}(0)$, then, for any direction $z \epsilon E^k$, there exists a vector $\bar{y} \epsilon E^{n-p}$ satisfying:

$$\text{i)} \quad -\nabla g_i(x^*,0) \, \bar{y} \leq z \nabla_\epsilon g_i(x^*,0) \quad \text{for } i \epsilon \widetilde{B}(0), \text{ and} \tag{3.6}$$

$$\text{ii)} \quad \nabla f(x^*,0) \, \bar{y} = \max_{\mu \epsilon \widetilde{K}(x^*,0)} \, [-z \, \nabla_\epsilon' g(x^*,0)\mu]. \tag{3.7}$$

<u>Proof</u>. Given $z \epsilon E^k$, consider the following linear program:

$$\max_\mu \, [-z \, \nabla_\epsilon' g(x^*,0)\mu]$$

$$\text{s.t.} \quad \mu \nabla g(x^*,0) = \nabla f(x^*,0)$$

$$\mu_i g_i(x^*,0) = 0 \quad (i=1,\ldots,m)$$

$$\mu_i \geq 0 \quad (i=1,\ldots,m).$$

The dual of this program is given by:

$$\min_{y,v} \quad \nabla f(x^*,0) \, y$$

$$\text{s.t.} \quad \nabla g_i(x^*,0) \, y + g_i(x^*,0) v_i \geq -z \, \nabla_\varepsilon g_i(x^*,0) \quad (i=1,\ldots,m)$$

$$y \varepsilon E^{n-p}, \; v_i \text{ unrestricted.}$$

Since MFCQ is assumed to hold at $(x^*,0)$, from [11] we have that $\widetilde{K}(x^*,0)$ is nonempty, compact and convex. Thus, the primal problem is bounded and feasible. By the duality theorem of linear programming, the dual program has a solution, $(\bar{y},\bar{v})$, and hence there exists a vector $\bar{y}$ satisfying (3.6) and (3.7).

By changing the sign of the objective function in the primal program above, thereby maximizing $z \, \nabla'_\varepsilon g(x^*,0)\mu$, we are also able to conclude the following.

<u>Theorem 3.4.</u> If the conditions of MFCQ are satisfied for some $x^* \varepsilon \widetilde{S}(0)$, then for any direction $z \varepsilon E^k$, there exists a vector $\bar{y} \varepsilon E^{n-p}$ satisfying:

i) $\nabla g_i(x^*,0) \, \bar{y} \geq z \nabla_\varepsilon g_i(x^*,0)$ for $i \varepsilon \widetilde{B}(0)$, and $\qquad\qquad$ (3.8)

ii) $\nabla f(x^*,0) \, \bar{y} = \max_{\mu \varepsilon \widetilde{K}(x^*,0)} [z \nabla'_\varepsilon g(x^*,0)\mu].$ $\qquad\qquad$ (3.9)

By taking $f(x,0) \equiv$ constant, then since any point in $\widetilde{R}(0)$ solves $\widetilde{P}(0)$, it easily follows from Theorems 3.3 and 3.4 that there exists a vector $\bar{y}$ satisfying (3.6) and (3.8) for any x^* in $\widetilde{R}(0)$.

In the next two theorems we show first that, along any ray emitting from $\varepsilon = 0$, $\widetilde{P}(\varepsilon)$ has points of feasibility near $\varepsilon = 0$, and second, that the existence of feasible points is guaranteed not only along rays but in a full neighborhood of $\varepsilon = 0$.

Theorem 3.5. Let $g: E^{n-p} \times E^k \rightarrow E^m$, with $g \varepsilon C^1$. If MFCQ holds at $x^* \varepsilon \widetilde{R}(0)$ then, for any unit vector $z \varepsilon E^k$ and any $\delta > 0$, $g(x^* + \beta(\bar{y} + \delta \widetilde{y}), \beta z) > 0$ for β positive and *sufficiently near zero*, where $\bar{y}$ is any vector satisfying (3.6).

Proof. Let z be any unit vector in E^k and consider first the case in which the constraint $g_i(x, \varepsilon) \geq 0$ is binding at $(x^*, 0)$. Expanding $g_i(x^* + \beta(\bar{y} + \delta \widetilde{y}), \beta z)$ about the point $(x^*, 0)$ we obtain:

$$g_i(x^* + \beta(\bar{y} + \delta \widetilde{y}), \beta z) = \beta(\bar{y} + \delta \widetilde{y}) \, \nabla_x g_i(x^* + t\beta(\bar{y} + \delta \widetilde{y}), \beta z)$$

$$+ \beta \, z \, \nabla_\varepsilon g_i(x^*, t'\beta z)$$

$$= \beta[\bar{y} \, \nabla_x g_i(x^* + t\beta(\bar{y} + \delta \widetilde{y}), \beta z) + z \, \nabla_\varepsilon g_i(x^*, t'\beta z)]$$

$$+ \beta \delta \widetilde{y} \, \nabla_x g_i(x^* + t\beta(\bar{y} + \delta \widetilde{y}), \beta z),$$

where $t, t' \varepsilon (0,1)$ and $t = t(\beta)$, $t' = t'(\beta)$.

Now, by (2.1), $\nabla_x g_i(x^*, 0) \, \widetilde{y} = a_i > 0$. Thus, there exists $\beta' > 0$ such that for all $\beta \varepsilon [0, \beta']$,

$$\widetilde{y} \, \nabla_x g_i(x^* + t\beta(\bar{y} + \delta \widetilde{y}), \beta z) \geq \frac{3a_i}{4} .$$

From (3.6) it follows that for β sufficiently small,

$$\bar{y} \, \nabla_x g_i(x^* + t\beta(\bar{y} + \delta \widetilde{y}), \beta z) + z \, \nabla_\varepsilon g_i(x^*, t'\beta z) \geq - \frac{\delta a_i}{4}.$$

Thus, for β positive and near zero we have:

$$g_i(x^* + \beta(\bar{y} + \delta\tilde{y}), \beta z) \geq \beta\left(-\frac{\delta a_i}{4}\right) + \beta\delta\left(\frac{3a_i}{4}\right) = \frac{\beta\delta a_i}{2} > 0.$$

Finally, if $g_i(x^*, 0) > 0$, since each g_i is jointly continuous in x and ε, it follows that, for any unit vector $z\varepsilon E^k$, and any $\delta > 0$
$g_i(x^* + \beta(\bar{y} + \delta\tilde{y}), \beta z) > 0$ for β near zero.

<u>Theorem 3.6</u>. If MFCQ is satisfied at $x^* \varepsilon \tilde{R}(0)$, then there exists $\bar{\beta} > 0$ such that for every unit vector $z\varepsilon E^k$ and any $\delta > 0$,
$g(x^* + \beta(\bar{y} + \delta\tilde{y}), \beta z) > 0$ for all $\beta\varepsilon(0, \bar{\beta}]$, where $\bar{y}$ satisfies (3.6) and $\tilde{y}$ is given by the constraint qualification.

<u>Proof</u>. From the previous result, we have that for any unit vector $z\varepsilon E^k$ and any $\delta > 0$, there exists $\beta' = \beta'(z, \delta, \bar{y}, \tilde{y})$ such that for all $\beta\varepsilon(0, \beta']$, $g(x^* + \beta(\bar{y} + \delta\tilde{y}), \beta z) > 0$. For the remainder of this proof we suppress all but the first argument of β' writing $\beta' = \beta'(z)$, and further assume that $\beta'(z) = \sup \{\gamma : g(x^* + \beta(\bar{y} + \delta\tilde{y}), \beta z) > 0$ for all $\beta\varepsilon(0, \gamma)\}$.

Suppose there is no value $\bar{\beta}$ which satisfies the stated condition for all unit vectors z. Then there must be a sequence $\{z_n\}$ of unit vectors from E^k with $\beta'(z_n) \to 0$. By the compactness of the unit sphere, there exists a unit vector $\bar{z}\varepsilon E^k$ with $z_{n_j} \to \bar{z}$ for some $\{z_{n_j}\} \subseteq \{z_n\}$. Since $\beta'(z_n) \to 0$, there is an integer N such that for all $n \geq N$, $\beta'(z_n) \leq \beta'(\bar{z}) - \xi$ where $0 < \xi < \beta'(\bar{z})$. Relabel the subsequence $\{z_n : n \geq N\}$ so that it is indexed by $n = 1, 2, \ldots$; we now refer

to this subsequence as the sequence $\{z_n\}$. By the definition of $\beta'(z)$, we must have, for some $i = 1,\ldots,m$, $g_i(x^* + \beta(\bar{y} + \delta\tilde{y}),\beta z_n) \leq 0$ for $\beta\epsilon[\beta'(z_n), \eta)$ for some $\eta = \eta(z_n)$. However, since $z_n \to \bar{z}$ and g_i is continuous, there must be an integer $N_1 > N$ such that $g_i(x^* + \beta(\bar{y} + \delta\tilde{y}),\beta z_n) > 0$ for all $n \geq N_1$ and all $\beta\epsilon(0,\beta'(\bar{z}))$. Since $\beta'(z_n) < \beta'(\bar{z})$ for all $n \geq N$, we have a contradiction and the proof of the assertion is complete.

Since, by Theorem 3.3 and the observation immediately following Theorem 3.4, the satisfaction of the Mangasarian-Fromovitz constraint qualification at a feasible point, x^*, of $\tilde{P}(0)$ is enough to guarantee the existence of feasible points for $\tilde{P}(\epsilon)$ near x^*, one might suspect that there exist points feasible to $\tilde{P}(\epsilon)$ which are also feasible to $\tilde{P}(0)$. This is indeed the case as the next theorem implies (see the statement immediately following the proof of Theorem 3.7). We shall need Theorem 3.7 in obtaining one of the key results in Theorem 4.3.

<u>Theorem 3.7.</u> Let $\beta_n \to 0^+$ in E^1, let z be any unit vector in E^k, and let $\delta > 0$. If $x_n \epsilon\tilde{R}(\beta_n z)$, with $x_n \to x^* \epsilon\tilde{R}(0)$, and if the conditions of MFCQ are satisfied at x^*, then $x_n + \beta_n(\bar{y} + \delta\tilde{y})\epsilon\tilde{R}(0)$ for n sufficiently large, where $\bar{y}$ satisfies (3.8) and $\tilde{y}$ is given by the constraint qualification.

<u>Proof.</u> Consider first the case that $i\epsilon\tilde{B}(0)$. Expanding $g_i(x_n + \beta_n(\bar{y} + \delta\tilde{y}),0)$ about the point $(x_n,\beta_n z)$, we obtain:

$$g_i(x_n + \beta_n(\bar{y} + \delta\tilde{y}),0) = g_i(x_n,\beta_n z) + \beta_n(\bar{y} + \delta\tilde{y})\nabla g_i(x_n + t\beta_n(\bar{y} + \delta\tilde{y}),0)$$

$$- \beta_n z \, \nabla_\varepsilon g_i(x_n + \beta_n(\bar{y} + \delta\tilde{y}),t'\beta_n z),$$

where $t,t'\varepsilon(0,1)$, $t = t(\beta_n)$, $t' = t'(\beta_n)$. If, for n large, $x_n + \beta_n(\bar{y} + \delta\tilde{y}) \notin \tilde{R}(0)$, since x_n is feasible for $\tilde{P}(\beta_n z) \geq 0$, it must be that

$$\beta_n(\bar{y} + \delta\tilde{y}) \, \nabla g_i(x_n + t\beta_n(\bar{y} + \delta\tilde{y}),0) < \beta_n z \, \nabla_\varepsilon g_i(x_n + \beta_n(\bar{y} + \delta\tilde{y}),t'\beta_n z). \quad (3.10)$$

Dividing by β_n in (3.10) and taking the limit as $n \to \infty$ we have

$$(\bar{y} + \delta\tilde{y}) \, \nabla g_i(x^*,0) < z \, \nabla_\varepsilon g_i(x^*,0).$$

But this contradicts (3.8), since $\delta > 0$ and by MFCQ $\tilde{y} \, \nabla g_i(x^*,0) > 0$.

If, on the other hand, $i \notin \tilde{B}(0)$, $g_i(x_n + \beta_n(\bar{y} + \delta\tilde{y}),0) > 0$ for large n by the continuity of g_i and the fact that $x_n \to x^*$ and $\beta_n \to 0$. Thus, $x_n + \beta_n(\bar{y} + \delta\tilde{y}) \varepsilon \tilde{R}(0)$ for n sufficiently large.

It may be interesting to note, that by taking $x_n = x^* + \beta_n(\bar{y} + \delta\tilde{y})$ for each n in the hypothesis of Theorem 3.7, then Theorems 3.3, 3.4, 3.6, and 3.7 together imply that $\tilde{R}(\varepsilon)$ and $\tilde{R}(0)$ have points in common for ε near 0.

We now show that the optimal value function $f^*(\varepsilon)$ is continuous near $\varepsilon = 0$ under the given assumptions. This was proved by Fiacco [8], the details being repeated here to make this paper complete. This result will be needed in the proof of Theorem 4.3.

Lemma 3.8. If $\widetilde{R}(0)$ is nonempty and $\widetilde{R}(\epsilon)$ is uniformly compact for ϵ near zero, then $\widetilde{R}(\epsilon)$ is a u.s.c. mapping at $\epsilon = 0$.

Proof. Let $\epsilon_n \to 0$ in E^k, and (for n sufficiently large), let $x_n \epsilon \widetilde{R}(\epsilon_n)$. Thus, for n sufficiently large, $g(x_n, \epsilon_n) \geq 0$. By the uniform compactness of $\widetilde{R}(\epsilon)$, there exists a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_{n_j} \to \bar{x}$ for some $\bar{x}$ in the closure of $\bigcup\limits_{\epsilon \text{ in } N(0)} \widetilde{R}(\epsilon)$, where $N(0)$ is some neighborhood of $\epsilon = 0$. But by the continuity of g, we must have $0 \leq \lim\limits_{j \to \infty} g(x_{n_j}, \epsilon_{n_j}) = g(\bar{x}, 0)$. Thus $\bar{x} \epsilon \widetilde{R}(0)$ and we have that $\widetilde{R}(\epsilon)$ is u.s.c. at $\epsilon = 0$.

Theorem 3.9. If $\widetilde{R}(0)$ is nonempty, $\widetilde{R}(\epsilon)$ is uniformly compact for ϵ near zero, and if the conditions of MFCQ hold at $x^* \epsilon \widetilde{S}(0)$, then $f^*(\epsilon)$ is continuous at $\epsilon = 0$.

Proof. Let $\epsilon_n \to 0$ in E^k, and (for n sufficiently large) let $x_n \epsilon \widetilde{S}(\epsilon_n)$. By the uniform compactness of $\widetilde{R}(\epsilon)$, the sequence $\{x_n\}$ admits a convergent subsequence $\{x_{n_j}\}$. Let $\bar{x}$ denote the limit of that subsequence. From Lemma 3.8, $\widetilde{R}(\epsilon)$ is a u.s.c. mapping at $\epsilon = 0$, so $\bar{x} \epsilon \widetilde{R}(0)$. Thus,

$$\varliminf_{\epsilon \to 0} f^*(\epsilon) = \lim_{j \to \infty} f^*(\epsilon_{n_j}) = \lim_{j \to \infty} f(x_{n_j}, \epsilon_{n_j}) = f(\bar{x}, 0) \geq f^*(0),$$

and we see that $f^*(\epsilon)$ is lsc at $\epsilon = 0$.

Now let $\delta > 0$, let $\hat{y}$ be given by MFCQ for x^*, and let $\bar{y}$ satisfy (3.6). From Theorem (3.5) we know that $x^* + \beta_k(\bar{y} + \delta\hat{y}) \epsilon \widetilde{R}(\beta_k z)$

for all unit vectors $z \varepsilon E^k$ providing β_k is sufficiently near zero.
Letting $\varepsilon_k \to 0$ and without loss of generality, assuming $\varepsilon_n \neq 0$ for all
n, setting $\beta_k = ||\varepsilon_k||$ and $z_k = \varepsilon_k / ||\varepsilon_k||$, we see that

$$\overline{\lim_{\varepsilon \to 0}} \ f^*(\varepsilon) = \overline{\lim_{\varepsilon_k \to 0}} \ f^*(\varepsilon_k) \leq \overline{\lim_{k \to \infty}} \ f(x^* + \beta_k(\bar{y} + \delta \bar{y}), \beta_k z_k) = f(x^*, 0).$$

Thus $f^*(\varepsilon)$ is also usc at $\varepsilon = 0$ and we may conclude that $f^*(\varepsilon)$ is
continuous at $\varepsilon = 0$.

We should mention that the continuity of f^* requires only the
continuity of f in addition to the once (joint) continuous differ-
entiability of the constraints.

4. Bounds on the Parametric Variation of the Optimal Value Function

In this section we are concerned with the directional derivative
of the optimal value function for $P(\varepsilon)$. We first derive upper and
lower bounds on the directional derivative limit quotient of $f^*(\varepsilon)$
for $\widetilde{P}(\varepsilon)$ and then obtain the corresponding bounds for $P(\varepsilon)$. These
results extend the work of Gauvin and Tolle [11], who obtained the
analogous results for the case in which the perturbation is restricted
to the right-hand side of the constraints.

As above, we will, without loss of generality, focus attention
on the parameter value $\varepsilon = 0$. For $z \varepsilon E^k$, the directional derivative
of $f^*(0)$ in the direction z is defined to be:

$$D_z f^*(0) = \lim_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \ , \qquad (4.1)$$

providing that the limit exists.

<u>Theorem 4.1</u>. If, for $\widetilde{P}(\varepsilon)$, MFCQ holds for some $x^* \varepsilon \widetilde{S}(0)$, then, for any direction $z \varepsilon E^k$,

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \max_{\mu \varepsilon \widetilde{K}(x^*,0)} z \nabla_\varepsilon \widetilde{L}(x^*,\mu,0). \qquad (4.2)$$

<u>Proof</u>. Let β satisfy the conditions of Theorem 3.6, let $\delta > 0$ and $\widehat{y}$ be the vector given by the constraint qualification, and let $\bar{y}$ satisfy eqs. (3.6) and (3.7). Then, for any $z \varepsilon E^k$, $x^* + \beta(\bar{y} + \delta \widehat{y}) \varepsilon \widetilde{K}(\beta z)$ for $\beta \varepsilon [0,\bar{\beta}]$ for some $\bar{\beta} > 0$, so that

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \limsup_{\beta \to 0^+} \frac{f(x^* + \beta(\bar{y} + \delta \widehat{y}), \beta z) - f(x^*,0)}{\beta} = \frac{df}{d\beta}(x^*,0)$$

$$= (\bar{y} + \delta \widehat{y}) \nabla f(x^*,0) + z \nabla_\varepsilon f(x^*,0).$$

Since this inequality is satisfied for arbitrary $\delta > 0$ we can take the limit as $\delta \to 0$ and obtain:

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \bar{y} \ \nabla f(x^*,0) + z \nabla_\varepsilon f(x^*,0).$$

The conclusion now follows by applying (3.7):

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \max_{\mu \epsilon \widetilde{K}(x^*,0)} [-z\nabla'_\epsilon g(x^*,0)\mu] + z\nabla_\epsilon f(x^*,0)$$

$$= \max_{\mu \epsilon \widetilde{K}(x^*,0)} z\nabla_\epsilon \widetilde{L}(x^*,\mu,0).$$

Corollary 4.2. Under the hypotheses of the previous theorem, if MFCQ holds at each point $x\epsilon \widetilde{S}(0)$, then

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \inf_{x\epsilon \widetilde{S}(0)} \max_{\mu \epsilon \widetilde{K}(x,0)} z\nabla_\epsilon \widetilde{L}(x,\mu,0). \qquad (4.3)$$

Proof. The result follows directly by applying the previous theorem at each point of $\widetilde{S}(0)$.

Theorem 4.3. If, for $\widetilde{P}(\epsilon)$, $\widetilde{R}(0)$ is nonempty, $\widetilde{R}(\epsilon)$ is uniformly compact near $\epsilon = 0$, and MFCQ holds for each $x^*\epsilon \widetilde{S}(0)$, then, for any direction $z\epsilon E^k$, z a unit vector,

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \ge \min_{\mu \epsilon \widetilde{K}(x^*,0)} z\nabla_\epsilon \widetilde{L}(x^*,\mu,0) \qquad (4.4)$$

holds for some $x^*\epsilon \widetilde{S}(0)$.

Proof. Let $x_n \epsilon \widetilde{S}(\beta_n z)$ and let $\beta_n \to 0^+$ be such that

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} = \lim_{n \to \infty} \frac{f(x_n,\beta_n z) - f(x^*,0)}{\beta_n} .$$

Since $R(\epsilon)$ is uniformly compact, there exists a subsequence, which we again denote by $\{x_n\}$, and a vector x^* such that $x_n \to x^*$. By Lemma 3.8 $\widetilde{R}(\epsilon)$ is u.s.c. at $\epsilon = 0$ so $x^* \epsilon \widetilde{R}(0)$. Since we showed in Theorem 3.9 that $f^*(\epsilon)$ is continuous, it follows that $x^* \epsilon \widetilde{S}(0)$. Then, since Theorem 3.7 assures that $x_n + \beta_n(\bar{y} + \delta\tilde{y}) \epsilon \widetilde{R}(0)$ for n sufficiently large, it follows that

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} = \lim_{n \to \infty} \frac{f(x_n, \beta_n z) - f(x^*, 0)}{\beta_n}$$

$$\geq \lim_{n \to \infty} \frac{f(x_n, \beta_n z) - f(x_n + \beta_n(\bar{y} + \delta\tilde{y}), 0)}{\beta_n}$$

$$= \lim_{n \to \infty} [-(\bar{y} + \delta\tilde{y})\nabla f(\alpha_n) + z\nabla_\epsilon f(\alpha_n)]$$

by the mean value theorem, where α_n is the usual convex combination (in $E^{n-p} \times E^k$) of the two arguments in the preceding quotient. Thus,

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \lim_{n \to \infty} z \nabla_\epsilon f(\alpha_n) - (\bar{y} + \delta\tilde{y}) \nabla f(\alpha_n)$$

$$= z \nabla_\epsilon f(x^*, 0) - (\bar{y} + \delta\tilde{y}) \nabla f(x^*, 0).$$

Using (3.9) and noting that δ was chosen as any positive value, we conclude that for any $x^* \epsilon \widetilde{S}(0)$ where MFCQ holds,

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq z \nabla_\epsilon f(x^*, 0) - \max_{\mu \epsilon \widetilde{R}(x^*, 0)} [z\nabla_\epsilon g(x^*, 0)\mu]$$

$$= \min_{\mu \epsilon \widetilde{R}(x^*, 0)} z \nabla_\epsilon \widetilde{L}(x^*, \mu, 0).$$

<u>Corollary 4.4.</u> Under the hypotheses of the previous theorem

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \inf_{x \in \widetilde{S}(0)} \min_{\mu \in \widetilde{K}(x,0)} z \nabla_\varepsilon \widetilde{L}(x,\mu,0). \qquad (4.5)$$

Now, by the reduction of variables that was applied earlier, we see that, in a neighborhood of $(x_I^*,0)$, with $x = (x_D(x_I,\varepsilon),x_I)$, that

$$L(x,\mu,\omega,\varepsilon) = f(x,\varepsilon) - \mu g(x,\varepsilon) + \omega h(x,\varepsilon)$$

$$= f(x_D(x_I,\varepsilon),x_I,\varepsilon) - \mu g(x_D(x_I,\varepsilon),x_I,\varepsilon) + \omega h(x_D(x_I,\varepsilon),x_I,\varepsilon),$$

$$= \widetilde{f}(x_I,\varepsilon) - \mu \widetilde{g}(x_I,\varepsilon) = \widetilde{L}(x_I,\mu,\varepsilon)$$

with $f(x,\varepsilon) \equiv \widetilde{f}(x_I,\varepsilon)$, $g(x,\varepsilon) \equiv \widetilde{g}(x_I,\varepsilon)$, and $h(x,\varepsilon) \equiv \widetilde{h}(x_I,\varepsilon) \equiv 0$. Thus $L(x,\mu,\omega,\varepsilon) \equiv \widetilde{L}(x_I,\mu,\varepsilon)$ in a neighborhood of $(x_D(x_I^*,0),x_I^*,0) = (x^*,0)$ and, with ω determined by $\omega = -(\nabla_{x_D} f - \mu\nabla'_{x_D} g)[\nabla'_{x_D} h^{-1}]$, it follows easily that $\nabla_\varepsilon \widetilde{L} = \nabla_\varepsilon L$ and the linear program appearing in the proof of Theorem 3.3 can readily be formulated analogously as an equivalent problem in terms of $L(x,\mu,\omega,\varepsilon)$. Thus, all of the results obtained above for $\widetilde{P}(\varepsilon)$ can be immediately generalized to $P(\varepsilon)$. For completeness, we state these results as the next Theorem.

<u>Theorem 4.5.</u> If, for $P(\varepsilon)$, $R(0)$ is nonempty and MFCQ holds at each $x \in S(0)$, then for any unit vector $z \in E^k$,

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \inf_{x \in S(0)} \max_{(\mu,\omega) \in K(x,0)} z \nabla_\varepsilon L(x,\mu,\omega,0), \qquad (4.6)$$

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and if $R(\varepsilon)$ is uniformly compact for ε near $\varepsilon = 0$, then

$$\lim_{\beta \to 0^+} \inf \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \inf_{x \varepsilon S(0)} \min_{(\mu,\omega) \varepsilon K(x,0)} z \, \nabla_\varepsilon L(x,\mu,\omega,0). \qquad (4.7)$$

Moreover, we are able to obtain the existence of the directional derivative of f^* at $\varepsilon = 0$ by assuming, as Gauvin and Tolle [11] did for right-hand side programs, the linear independence of the binding constraint gradients at each point $x^* \varepsilon S(0)$.

Corollary 4.6. Assume $R(0)$ is nonempty and $R(\varepsilon)$ is uniformly compact near $\varepsilon = 0$. If the gradients, taken with respect to x, of the constraints binding at x^* are linearly independent for each $x^* \varepsilon S(0)$, then for any unit vector $z \varepsilon E^k$, $D_z f^*(0)$ exists and is given by

$$D_z f^*(0) = \inf_{x \varepsilon S(0)} z \, \nabla_\varepsilon L(x,\mu(x),\omega(x),0),$$

where $(\mu(x),\omega(x))$ is the unique multiplier vector associated with x.

Proof. At any point $x^* \varepsilon S(0)$, the linear independence of the binding constraint gradients implies the uniqueness of the Kuhn-Tucker multipliers corresponding to x^*. Inequalities (4.6) and (4.7) now combine to yield the desired result.

Note that in Corollary 4.6, if $P(\varepsilon)$ contains no inequality constraints, we could replace inf by min since μ would not appear and $\omega = - \nabla_{x_D} f \, \nabla_{x_D} h^{-1}$ which is continuous in x, making $z \nabla_\varepsilon L$ a continuous function of x minimized over $S(0)$, a compact set.

We may also show that two of the observations made by Gauvin and Tolle [11] about $D_z f^*(0)$ for right-hand side programs apply to $P(\varepsilon)$

as well. First if $D_z f^*(0) = - D_{-z} f^*(0)$, then

$$\inf_{x \in S(0)} \max_{(\mu,\omega) \in K(x,0)} z \nabla_\epsilon L(x,\mu,\omega,0) = \sup_{x \in S(0)} \min_{(\mu,\omega) \in K(x,0)} z \nabla_\epsilon L(x,\mu,\omega,0). \quad (4.8)$$

Thus, if, for all unit vectors $z \epsilon E^k$, $D_z f^*(0) = - D_{-z} f^*(0)$ and

$$D_z f^*(0) = \inf_{x \in S(0)} \max_{(\mu,\omega) \in K(x,0)} z \nabla_\epsilon L(x,\mu,\omega,0), \quad (4.9)$$

then (4.8) provides a necessary condition for the existence of $\nabla_\epsilon f^*(0)$. In addition, if (4.8) holds for every unit vector $z \epsilon E^k$ and if $x^* \epsilon S(0)$ is the unique solution of $P(0)$, then its associated Kuhn-Tucker multiplier vector is unique.

We next apply the results derived above to a particular class of programs. We show in the next theorem that if $P(\epsilon)$ is a convex program in x for ϵ near $\epsilon = 0$, i.e., if $f(x,\epsilon)$ and $-g_i(x,\epsilon)$, $i = 1,\ldots,m$, are convex and if $h_j(x,\epsilon)$, $j = 1,\ldots,p$, are affine in x, then $D_z f^*(0)$ exists and is given by (4.9). To prove this result, we will restrict our attention to $\widetilde{P}(\epsilon)$. We are able to do this since the functions $h_j(x,\epsilon)$ are assumed to be affine in x and $g_i(x,\epsilon)$ are taken to be concave in x, from which it easily follows that $\widetilde{g}_i(x,\epsilon)$ is concave in x for $i = 1,\ldots,m$.

<u>Theorem 4.7</u>. In $P(\epsilon)$, let $f(x,\epsilon)$ and $-g_i(x,\epsilon)$, $i = 1,\ldots,m$ be convex and let $h_j(x,\epsilon)$, $j = 1,\ldots,p$ be affine in x. If $R(0)$ is nonempty, $R(\epsilon)$ is uniformly compact near $\epsilon = 0$, and MFCQ holds for each $x^* \epsilon S(0)$, then, for any unit vector $z \epsilon E^k$,

$$D_z f^*(0) = \inf_{x \in S(0)} \max_{(\mu,\omega) \in K(x,0)} z \nabla_\epsilon L(x,\mu,\omega,0). \quad (4.10)$$

Proof. Without loss of generality, we will prove this result for $\widetilde{P}(\varepsilon)$. Let $x^*\varepsilon\widetilde{S}(0)$ and $x_n\varepsilon\widetilde{S}(\beta_n z)$ with $\beta_n \to 0^+$ in such a way that

$$\lim_{\beta\to 0^+} \inf \frac{f^*(\beta z) - f^*(0)}{\beta} = \lim_{n\to\infty} \frac{f(x_n,\beta_n z) - f(x^*,0)}{\beta_n} ,$$

and $x_n \to x^*$ as in the proof of Theorem 4.3. For all $\mu^*\varepsilon\widetilde{K}(x^*,0)$,

$$\widetilde{L}(x_n,\mu^*,\beta_n z) = f(x_n,\beta_n z) - \mu^* g(x_n,\beta_n z) \leq f(x_n,\beta_n z),$$

where the inequality follows from the non-negativity of both μ^* and $g(x_n,\beta_n z)$. Thus, since $\widetilde{L}(x^*,\mu^*,0) = f(x^*,0)$,

$$\lim_{n\to\infty} \frac{f(x_n,\beta_n z) - f(x^*,0)}{\beta_n} \geq \lim_{n\to\infty} \frac{\widetilde{L}(x_n,\mu^*,\beta_n z) - \widetilde{L}(x^*,\mu^*,0)}{\beta_n} .$$

Now, as a result of the Kuhn-Tucker conditions and the convexity assumptions, x^* is a global minimizer of $\widetilde{L}(x,\mu^*,0)$, so

$$\lim_{n\to\infty} \frac{\widetilde{L}(x_n,\mu^*,\beta_n z) - \widetilde{L}(x^*,\mu^*,0)}{\beta_n} \geq \lim_{n\to\infty} \frac{\widetilde{L}(x_n,\mu^*,\beta_n z) - \widetilde{L}(x_n,\mu^*,0)}{\beta_n}$$

$$= \lim_{n\to\infty} \frac{\widetilde{L}(x_n,\mu^*,0) + \beta_n z \nabla_\varepsilon \widetilde{L}(x_n,\mu^*,t\beta_n z) - L(x_n,\mu^*,0)}{\beta_n}$$

by the mean value theorem, where $t\varepsilon(0,1)$. Thus

$$\lim_{n\to\infty} \frac{f(x_n,\beta_n z) - f(x^*,0)}{\beta_n} \geq \lim_{n\to\infty} z \nabla_\varepsilon \widetilde{L}(x_n,\mu^*,t\beta_n z),$$

and, passing to the limit on the right, we are able to conclude that

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$$\lim_{n\to\infty} \frac{f(x_n,\beta_n z) - f(x^*,0)}{\beta_n} \geq z\nabla_\epsilon \widetilde{L}(x^*,\mu^*,0). \qquad (4.11)$$

Thus, for some $x^*\epsilon\widetilde{S}(0)$, since (4.11) holds for each $\mu^*\epsilon\widetilde{K}(x^*,0)$, and recalling from [11] that $\widetilde{K}(x^*,0)$ is compact,

$$\liminf_{\beta\to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \max_{\mu\epsilon\widetilde{K}(x^*,0)} z\nabla_\epsilon \widetilde{L}(x^*,\mu,0),$$

from which we see that

$$\liminf_{\beta\to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \inf_{x\epsilon\widetilde{S}(0)} \max_{\mu\epsilon\widetilde{K}(x^*,0)} z\nabla_\epsilon \widetilde{L}(x,\mu,0).$$

Combining this result with that obtained in Corollary 4.2 we conclude that

$$D_z f^*(0) = \inf_{x\epsilon\widetilde{S}(0)} \max_{\mu\epsilon\widetilde{K}(x,0)} z\nabla_\epsilon \widetilde{L}(x,\mu,0).$$

For convex $P(\epsilon)$, (4.10) now follows by an inversion of the reduction of variables process applied to yield $\widetilde{P}(\epsilon)$.

5. Example

We use the example stated below to demonstrate the theoretical results obtained in the previous sections. For the given problem we show that the conditions of MFCQ hold at every point in $S(\epsilon)$ and we give the form of the vector satisfying the constraint qualification. We then show that the expected form of the vector satisfies (2.1) and (2.2) for $\widetilde{P}(\epsilon)$, and that the bounds stated in (4.6) and (4.7) are attained.

Consider the program

$$\min \; \varepsilon x_1 \qquad\qquad\qquad P(\varepsilon)$$

$$\text{s.t. } g(x,\varepsilon) = - (x_1-\varepsilon)^2 - (x_2 \pm 2)^2 + 4 \geq 0$$

$$h(x,\varepsilon) = - x_1 + x_2 + \varepsilon = 0.$$

The solution of this program is easily determined to be $x_1^* = x_2^* + \varepsilon$ with

$$x_2^* = \begin{cases} 0 & \varepsilon > 0 \\ 2 & \varepsilon < 0 \end{cases}, \text{ and if } \varepsilon = 0, \; x_2^* \text{ can be any value in the interval } [0,2]. \quad (5.1)$$

Applying the reduction of variables technique outlined earlier, with $x_D = x_1$ and $x_I = x_2$, $P(\varepsilon)$ is transformed into the equivalent program

$$\min \quad \varepsilon(x_2 + \varepsilon) \qquad\qquad\qquad \widetilde{P}(\varepsilon)$$

$$\text{s.t. } \widetilde{g}(x_2,\varepsilon) = - x_2^2 - (x_2-2)^2 + 4 \geq 0$$

whose solution is given by (5.1).

For both $P(\varepsilon)$ and $\widetilde{P}(\varepsilon)$, the optimal value function can be written as

$$f^*(\varepsilon) = \begin{cases} \varepsilon^2 & \varepsilon \geq 0 \\ \varepsilon^2 + 2\varepsilon & \varepsilon < 0 \end{cases}. \qquad (5.2)$$

We see that f^* is continuous for all values of ε, but it is not differentiable at $\varepsilon = 0$. It does, however, have directional derivatives at $\varepsilon = 0$ which are given by

$$D_z f^*(0) = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases}. \qquad (5.3)$$

To illustrate Theorem 3.2, we first determine the general form of the vector, $\widetilde{y}$, which is guaranteed at points $x \epsilon S(\epsilon)$ at which MFCQ is satisfied. The constraint gradients of $P(\epsilon)$ are

$$\nabla g(x, \epsilon) = [-2(x_1 - \epsilon), \ -2(x_2 - 2)], \text{ and}$$

$$\nabla h(x, \epsilon) = [-1, 1].$$

Applying (2.1) and (2.2) at a point $x^* = (x_1^*, x_2^*) \epsilon S(\epsilon)$, with $\widetilde{y} = (y_1, y_2)$, we have $\nabla g(x^*, \epsilon) \ \widetilde{y} = -2(x_1^* - \epsilon) \ y_1 - 2(x_2 - 2) \ y_2 > 0$ if $g(x^*, 0) = 0$, and

$$\nabla h(x^*, \epsilon) \ \widetilde{y} = -y_1 + y_2 = 0.$$

Thus, for any value of ϵ, since $g(x, \epsilon)$ is binding only if $x_2^* \epsilon \{0, 2\}$, $\widetilde{y}$ can have the form

$$\widetilde{y} = \begin{cases} (a, a) & x_2^* = 0 \\ (b, b) & 0 < x_2^* < 2 \ , \\ (c, c) & x_2^* = 2 \end{cases} \tag{5.4}$$

for any real numbers a, b, c with $a > 0$, b arbitrary, and $c < 0$. We can also conclude that MFCQ holds at every solution of $P(\epsilon)$.

In a similar fashion, we see that, for $\widetilde{P}(\epsilon)$,

$$\nabla \widetilde{g}(x_2, \epsilon) = -2x_2 - 2(x_2 - 2),$$

so applying (2.1) we find that the vector $\widetilde{y}_I$ in the reduced program takes the same form as the second component of $\widetilde{y}$ in (5.4).

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Now

$$\nabla L(x,\mu,\omega,\epsilon) = [\epsilon + 2\mu(x_1-\epsilon) - \omega, \; 2\mu(x_2-2) + \omega],$$

so that at a solution $x^*\epsilon S(0)$ we must have $2\mu x_1^* - \omega = 0$ for $(\mu,\omega)\epsilon K(x^*,0)$. Then

$$\nabla_\epsilon L(x^*,\mu,\omega,0) = x_1^* \, ,$$

and, with $S(0) = \{x\epsilon E^2: \; x_1=x_2, \; x_2\epsilon[0,2]\}$,

$$\min_{x\epsilon S(0)} \; \max_{(\mu,\omega)\epsilon K(x,0)} z \, \nabla_\epsilon L(x,\mu,\omega,0) = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases} . \qquad (5.5)$$

Comparing (5.3) with (5.5) we see that (4.6) holds with equality.

Now, considering inequality (4.7), we first note that for any neighborhood $N(0)$ of $0\epsilon E^1$, the closure of the set $\{x\epsilon E^2: \; x = (x_2+\epsilon, \; x_2), \; x_2\epsilon[0,2], \; \epsilon \text{ in } N(0)\}$ is compact so $R(\epsilon)$ is uniformly compact for ϵ near $\epsilon = 0$. We calculate

$$\min_{x\epsilon S(0)} \; \min_{(\mu,\omega)\epsilon K(x,0)} z \, \nabla_\epsilon L(x,\mu,\omega,0) = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases},$$

and find that (4.7) also holds with equality.

An example is given in [11] which illustrates that (4.6) and (4.7) need not hold with equality.

6. Related Results

Inspection of the derivation of (4.6) and (4.7) reveals that the bounding term in these expressions, namely $z\nabla_\epsilon L(x,\mu,\omega,0)$, can be

viewed as the sum of two distinct expressions, one resulting from the variation of the objective function of $P(\epsilon)$ with respect to the parameter, the other deriving from the dependence of the region of feasibility on the parameter. The first of these terms is $z \, \nabla_\epsilon f(x,0)$ and is easily seen to result directly from the manipulation of the limit quotients in the proofs of Theorems 4.1 and 4.3. The second component, $z[-\nabla_\epsilon' g(x,0)\mu + \nabla_\epsilon' h(x,\epsilon)\omega]$, results from the assumption that MFCQ holds at points of $S(0)$. The conditions of MFCQ are invoked to enable us to conclude (3.7) and (3.9), as well as the existence of points feasible to $P(\epsilon)$ in a neighborhood of $\epsilon = 0$. Having made these observations, we are now able to discuss the relationships between the bounds provided here and results previously obtained by others. As we shall see, in particular instances in which the directional derivative of f^* is shown to exist, it is expressed as either a function of $\nabla_\epsilon f$ or a function of $\nabla_\epsilon g$ and $\nabla_\epsilon h$, or a combination of all of these terms, depending, as one would suspect, on where in $P(\epsilon)$ the parameter appears.

Danskin [5,6] provided a now well-known characterization of the directional derivative of the optimal value function of $P(\epsilon)$ in the case that the constraints are independent of a parameter. Under the conditions that the region of feasibility, $R(0)$, is compact, and $f(x,\epsilon)$ and $\nabla_\epsilon f(x,\epsilon)$ are continuous at $\epsilon = 0$, Danskin showed that

$$D_z f^*(0) = \min_{x \in S(0)} \; z \, \nabla_\epsilon f(x,0) \; . \tag{6.1}$$

Relating our hypotheses to Danskin's construct, we first note the equivalence of our assumption of the uniform compactness of $R(\varepsilon)$ for ε near $\varepsilon = 0$, and the assumption that the feasible region is compact if the constraints of $P(\varepsilon)$ do not depend on ε. To see this, one need only consider that, in this case, $R(\varepsilon) \equiv R(0)$ for all ε and apply Definition 2.3. In addition, when the feasible region is independent of ε, our development need not consider the perturbed point $x^* + \beta(\bar{y} + \delta\bar{y})$, but may be restricted to the point $x^* \varepsilon S(0)$. The proofs of Theorems 4.1 and 4.2 remain valid in this case by simply suppressing all reference to the dependence of the constraints on ε and by considering the unperturbed point x^* instead of $x^* + \beta(\bar{y} + \delta\bar{y})$. One is then led to conclude that, analogous to (4.6) and (4.7),

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \min_{x \varepsilon S(0)} z \, \nabla_\varepsilon f(x,0), \text{ and} \qquad (6.2)$$

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \min_{x \varepsilon S(0)} z \, \nabla_\varepsilon f(x,0), \qquad (6.3)$$

for any unit vector $z \varepsilon E^k$. Thus it follows that, under the stated conditions, namely the compactness of R and the continuity of $f(x,\varepsilon)$ and $\nabla_\varepsilon f(x,\varepsilon)$ at $\varepsilon = 0$, our results are consistent with those of Danskin in that they verify the existence of $D_z f^*(0)$ and show (from (6.2) and (6.3)) that it can be expressed as in (6.1).

Gauvin and Tolle [11] showed, for programs with right-hand side perturbations, i.e., for programs of the form

$$\min f(x)$$

$$\text{s.t. } g_i(x) \geq \epsilon_i \qquad (i=1,\ldots,m),$$

$$h_j(x) = \epsilon_{m+j} \qquad (j=1,\ldots,p),$$

P'(ε)

that although the directional derivative of f^* may not exist, its limit quotient can be bounded. In particular, they concluded that if MFCQ holds at each element of $S(0)$ and if $R(\epsilon)$ is uniformly compact for ϵ near $\epsilon = 0$, the following inequalities are satisfied:

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \inf_{x \in S(0)} \max_{(\mu,\omega) \in K(x,0)} \left(\sum_{i=1}^m \mu_i z_i - \sum_{j=1}^p \omega_j z_{m+j} \right), \quad (6.4)$$

and

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \inf_{x \in S(0)} \min_{(\mu,\omega) \in K(x,0)} \left(\sum_{i=1}^m \mu_i z_i - \sum_{j=1}^p \omega_j z_{m+j} \right). \quad (6.5)$$

Now, from (4.6) and (4.7) we see that the bounds we have given for the general program $P(\epsilon)$ reduce to those in (6.4) and (6.5) respectively for the more restrictive perturbations appearing in P'(ε).

In the case of convex programs, the existence of $D_z f^*(0)$ assured by Theorem 4.7, and its expression as (4.10), corresponds under slightly different assumptions, with results achieved by Gol'stein [12] and Hogan [15]. Theorem 4.7 is a direct extension to the general perturbed mathematical program of a result given by Gauvin and Tolle [11] for right-hand side programs.

In an as yet unpublished manuscript communicated to us by J. Gauvin, A. Auslender has apparently extended the results of Gauvin and Tolle [11] to problems involving non-differentiable functions. In

particular, the bounds noted by (6.4) and (6.5) are obtained for

right-hand side programs in which the problem functions are locally

Lipschitz and those defining the equality constraints are continuously

differentiable. We are currently studying this result to understand

the relationship between our assumptions and those of Auslender and

to determine if Auslender's result extends to the more general pro-

gram $P(\varepsilon)$.

REFERENCES

1. Armacost, R. L., and A. V. Fiacco, "NLP Sensitivity Analysis for RHS Perturbations: A Brief Survey and Second Order Extensions," Technical Paper T-334, Institute for Management Science and Engineering, The George Washington University, April 1976.

2. - - - - -, "Second-Order Parametric Sensitivity Analysis in NLP and Estimates by Penalty Function Methods," Technical Paper T-324, Institute for Management Science and Engineering, The George Washington University, December 1975.

3. - - - - -, "Sensitivity Analysis for Parametric Nonlinear Programming Using Penalty Methods," *Computers and Mathematical Programming*, National Bureau of Standards Special Publication 502, pp. 261-269.

4. Berge, C., *Topological Spaces*, trans. by E. M. Patterson, The MacMillan Company, New York, New York, 1963.

5. Danskin, J. M., *The Theory of Max-Min*, Springer-Verlag Inc., New York, New York, 1967.

6. - - - - -, "The Theory of Max-Min with Applications," *SIAM Journal of Applied Mathematics*, 14(4), 1966, pp. 641-665.

7. Evans, J. P., and F. J. Gould, "Stability in Nonlinear Programming," *Operations Research*, 18(1), 1970, pp. 107-118.

8. Fiacco, A. V., "On the Continuity of the Optimal Value Function," in preparation.

9. - - - - -, "Sensitivity Analysis for Nonlinear Programming Using Penalty Methods," *Mathematical Programming*, 10(3), 1976, pp. 287-311.

10. Fiacco, A. V., and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, John Wiley and Sons, Inc., New York, New York, 1968.

11. Gauvin, J., and J. W. Tolle, "Differential Stability in Nonlinear Programming," *SIAM Journal of Control and Optimization*, 15(2), 1977, pp. 294-311.

12. Gol'stein, E. G., *Theory of Convex Programming*, Translations of Mathematical Monographs, Vol. 36, American Mathematical Society, Providence, Rhode Island, 1972.

13. Greenberg, H. J., and W. P. Pierskalla, "Extensions of the Evans-Gould Stability Theorems for Mathematical Programs," *Operations Research*, 20, 1972, pp. 143-153.

14. Hogan, W., "The Continuity of the Perturbation Function of a Convex Program," *Operations Research*, 21(1), 1973, pp. 351-352.

15. - - - - -, "Directional Derivatives for Extremal Value Functions with Applications to the Completely Convex Case," *Operations Research*, 21(1), 1973, pp. 188-209.

16. - - - - -, *Optimization and Convergence for Extremal Value Functions Arising from Structured Nonlinear Programs*, Doctoral Dissertation, Western Management Science Institute, University of California at Los Angeles, September 1971.

17. - - - - -, "Point-to-Set Maps in Mathematical Programming," *SIAM Review*, 15(3), 1973, pp. 591-603.

18. Maurer, H., "A Sensitivity Result for Infinite Nonlinear Programming Problems; Part I: Theory," Mathematisches Institut der Universitat Wurzburg, Wurzburg, W. Germany, January 1977.

19. - - - - -, "A Sensitivity Result for Infinite Nonlinear Programming Problems; Part II: Applications to Optimal Control Problems," Mathematisches Institut der Universitat Wurzburg, Wurzburg, W. Germany, January 1977.

20. Robinson, S. M., "Perturbed Kuhn-Tucker Points and Rates of Convergence for a Class of Nonlinear Programming Algorithms," *Mathematical Programming*, 7(1), 1974, pp. 1-16.

21. Rockafellar, R. T., "Augmented Lagrange Multiplier Functions and Duality in Nonconvex Programming," *SIAM Journal of Control and Optimization*, 12(2), 1974, pp. 268-285.

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